Noise-induced coherence in bistable systems with multiple time delays

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We study the correlation properties of noise-driven bistable systems with multiple time-delay feedbacks. For small noisy perturbation and feedback magnitude, we derive the autocorrelation function and the power spectrum based on the two-state model with transition rates depending on the earlier states of the system. A comparison between the single and double time delays reveals that the auto correlation functions exhibit exponential decay with small undulation for the double time delays, in contrast with the remarkable oscillatory behavior at small time lags for the single time delay.

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Noise-induced phenomena such as stochastic and coherence resonance in nonlinear dynamical systems have recently been under intensive analytical and numerical investigation [1–3]. The mechanism that underlies such resonancelike behavior is the time scale matching between the statistically averaged escape time of the stochastic perturbation, which can be tuned by its intensity and some intrinsic time scale of the dynamical system or external periodic signals. Since the conditions for occurrence of the stochastic and coherence resonance are simple and robust, it is not surprising that SR and CR are observed and have applications in many natural and artificial systems, such as, the neural system [4], optical system [5], and other physical systems. More recently, the study of the noise-induced resonance phenomena has been extended to nonlinear dynamical systems with delayed feedbacks [6–12]. It is found that the interplay between the noise and time delay can generate very complex dynamical behaviors that are relevant in many processes such as biophysiological controls [13], signal transmission in biological and artificial neuronal networks [14–16]. A theoretic and numerical analysis on noise-induced dynamics in bistable systems with time-delayed feedback loops have been carried out, based on a two-state model system [17]. In Ref. [9], Tsimring and Pikovsky have developed a theory of a prototypical noise-driven bistable system with delayed feedback. For small noise and amplitude of the feedback, they derived the analytical formulas for the autocorrelation function and the power spectrum that are in a very good agreement with direct numerical simulations of the original Langevin equation.

In this paper, we address the important issue of noise-driven dynamics with multiple delays. We will derive a general theory for stochastic bistable system with multiple time-delayed feedback loops within the framework of the two-state system approximation. It is well known that the behavior of multistable dynamical systems with memory depends on its past through some memory kernels formed by multiple feedback loops with different delay times.

We consider the over-damped particle motion in a double-well quartic potential with a n-tuple time-delayed feedback, which is described by

$$\frac{dx}{dt} = x(t) - x(t)^3 + \sum_{j=1}^{n} \epsilon_j x(t - T_j) + \sqrt{2D}\xi(t). \tag{1}$$

Here T_j is the jth time delay and ϵ_j is the strength of the jth feedback. $\xi(t)$ is a Gaussian white noise with $\langle\xi\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=\delta(t-t')$. Without loss of generality we assume that $T_1 \leqslant T_2 \leqslant \cdots \leqslant T_n$. In the absence of delays, a damped particle will spend most of its time near the stable states located at $x=\pm1$, and make occasional transition over the barrier in the center, under a moderate amount of random forcing. Thus, when ϵ_j and D are small, the intrawell motion can be neglected and Eq. (1) can be approximated by a two-state system with its dynamical variable $s(t)=\pm1$, corresponding to $x>0$ and $x<0$, respectively. We denote $n_\pm$ to be the probabilities that the system occupies either state ± at time t. Then, the governing equation for $n_\pm$ is given by

$$\frac{dn_+}{dt} = -\frac{dn_-}{dt} = W_-(t) - [W_-(t) + W_+(t)]n_+, \tag{2}$$

where $W_\pm$ is the transition rate out of the ± state. The dynamics of the two-state system is determined by the switching rates $s \to -s$. Taking into account an n-tuple time delays, we have $N=2^n$ switching rates related to the states $s(t-T_i)$, $1 \leqslant i \leqslant n$. We define by $p(s_1,s_2,\ldots,s_n)$ the switching rate $s(t) \to -s(t)$ given that the n earlier states are in the state $s_i = s(t-T_i)$ with $1 \leqslant i \leqslant n$. Therefore, for our stochastic system with multiple delays the transition rates are given by

$$W_-(t) = \sum_{j=1}^{n} \sum_{s_j=\pm} p(s_1,s_2,\ldots,s_n) n_{s_1} n_{s_2} \cdots n_{s_n}, \tag{3}$$

$$W_+(t) = \sum_{j=1}^{n} \sum_{s_j=\pm} p(-s_1,-s_2,\ldots,-s_n) n_{s_1} n_{s_2} \cdots n_{s_n}, \tag{4}$$

where the sum is over $N=2^n$ combinations of $\{s_j=\pm1, j=1,2,\ldots,n\}$ and $n_{s_j}=n_{s_j}(t-T_j)$. To simplify the notation we introduce $a_l=p(s_1,s_2,\ldots,s_n)$ to represent the lth combination with $l=\sum_{i=1}^{n}(1-s_i^l)2^{n-i}$. The switching rates a_l can be determined from the original continuous bistable system (1). We assume that when ϵ_i and D are small, a_l is given by the Kramers escape rate $r_K=(2\pi)^{-1}\sqrt{U''(x_m)U''(x_0)}\exp(-\Delta U/D)$, where x_m and x_0 are the positions of the minima and maximum of the potential, respectively, and ΔU is the potential barrier. In the case of multiple time-delay feedbacks, the positions of the potential minima are approximated by $|x_m|=1+\epsilon^l$, with the effective feedback coupling given by $\epsilon^l=\sum_{j=1}^{n} s_j^l\epsilon_j$. Here s_j is determined by the sign of $s(t)s(t-T_j)$. For system (1) with small ϵ_i and D, we can write the modified Kramers escape rates as

$$a_l = \frac{\sqrt{2+3\epsilon^l}}{2\pi}e^{-(1+4\epsilon^l)/4D}. \tag{5}$$

The correlation function is defined as

$$C(\tau) = \langle s(\tau)s(0)\rangle = \langle s(\tau)\rangle = 2\langle n_+(\tau)\rangle - 1, \tag{6}$$

here we have made use of the normalization condition $n_+ + n_- = 1$, and the assumption that the system is initially in the state +, i.e., $\sigma(0)=1$. On substitution of Eq. (6) into Eq. (2), we find

$$\frac{dC(t)}{dt} = W_-(t) - W_+(t) - [W_+(t) + W_-(t)]C(t). \tag{7}$$

By inserting Eqs. (3) and (4) into the governing equation for the correlation function we find, after some algebra, the nonlinear equation for the correlation function

$$\frac{dC(t)}{dt} = -p_0C(t) + \sum_{j=1}^{n} p_jC(t-T_j) + \sum_{i,j=1} p_{i,j}C(t-T_i)C(t-T_j) + p_{i,j,k}C(t-T_i)C(t-T_j)C(t-T_k) + \cdots + p_{1,2,\ldots,n}\prod_{j=1}^{n} C(t-T_j). \tag{8}$$

The coefficients are found to be

$$p_0 = \sum_{l=1}^{n} a_l,$$

$$p_j = \sum_{l=1}^{n} (-1)^{m_{jl}}a_l,$$

$$p_{i,j} = \sum_{l=1}^{n} (-1)^{m_{il}+m_{jl}}a_l,$$

$$p_{i,j,k} = \sum_{l=1}^{n} (-1)^{m_{il}+m_{jl}+m_{kl}}a_l,$$

$$\cdots$$

$$p_{j_1j_2\cdots j_k} = \sum_{l=1}^{n} (-1)^{m_{j_1l}+m_{j_2l}+\cdots+m_{j_kl}}a_l,$$

$$\cdots$$

$$p_{1,2,\ldots,n} = \sum_{l=1}^{N} (-1)^{m_{1l}+m_{2l}+\cdots+m_{nl}}a_l, \tag{9}$$

where

$$m_{jl} = \left[\frac{l-1}{2^{n-j}}\right] \quad (1 \leqslant j,\ l \leqslant n) \tag{10}$$

are integers. Taking into account the weak delay feedbacks, i.e., $\epsilon_j \ll 1$, we expand $\sqrt{(2+3\epsilon)}$ and then find

$$a_l \approx \frac{e^{-(1+4\epsilon^l)/4D}}{\sqrt{2}\pi}\left(1 + \frac{3}{4}\sum_{j=1}^{n} s_j\epsilon_j\right). \tag{11}$$

It can be verified that to the first order in ϵ, the coefficients in front of the nonlinear terms of Eq. (8) are vanishing and the governing equation for $C(t)$ can be approximated by

$$\frac{dC(t)}{dt} = -p_0C(t) + \sum_{j=1}^{n} p_jC(t-T_j). \tag{12}$$

In comparison with the governing equation for the single time delay, the additional terms with different delay times may weaken or enhance the oscillatory nature of the autocorrelation function as shown in Ref. [9], depending on the phase of $C(t-T_j)$. Clearly if the effect of multiple time delays is vanishing due to the cancellation among the delayed feedbacks, the exponential decay will be a dominant feature of the correlation function.

In the following we discuss the solution to this linear equation under the normalization condition $C(0)=1$, and the symmetry requirement $C(-t)=C(t)$. In order to obtain the correlation function $C(t)$, it is necessary to know $C(t)$ at the time interval $[0,T_n]$. Because of the symmetry, we need only to compute $C(t)$ on the time interval $[0,T_n/2]$.

(i) On the interval $(0,T_1)$, we use the ansatz,

$$C(t) = A_0e^{-\lambda_0|t|} + B_0e^{\lambda_0|t|}, \tag{13}$$

which satisfies the symmetry requirement $C(-t)=C(t)$ by definition, and the normalization gives rise to $A_0+B_0=1$. By inserting this ansatz into Eq. (12) we find

$$\left[\sum_{i=1}^{n} p_ie^{\lambda_0T_i}\right]\left[\sum_{j=1}^{n} p_je^{-\lambda_0T_j}\right] = p_0^2 - \lambda_0^2 \tag{14}$$

and

$$A_0 = \frac{\sum_{j=1}^{n} p_je^{\lambda_0T_j}}{p_0 - \lambda_0 + \sum_{j=1}^{n} p_je^{\lambda_0T_j}}, \tag{15}$$

$$B_0 = \frac{p_0 - \lambda}{p_0 - \lambda_0 + \sum_{j=1}^{n} p_j e^{\lambda_0 T_j}}. \quad (16)$$

If $T_n/2 \leqslant T_1$, then we can go to solve the linear differential equation (12), with all the delay terms known. Otherwise we have to find the solution to Eq. (12) using a similar ansatz on each time intervals involved.

(ii) Suppose that $T_{k-1} \leqslant T_n/2 \leqslant T_k$, then on the interval sets $\{[T_i, T_{i+1}]: 2 \leqslant i \leqslant k-2\}$ we use the following ansatz:

$$C(t) = C(T_i)[A_i e^{-\lambda_i(|t|-T_i)} + B_i e^{\lambda_i(|t|-T_i)}]. \quad (17)$$

On substitution of the above ansatz into Eq. (12), we obtain

$$\left[p_0 - \lambda_i - \sum_{j=1}^{i} p_j e^{\lambda_i T_j}\right]\left[p_0 + \lambda_i - \sum_{j=i}^{n} p_j e^{-\lambda_i T_j}\right] = \sum_{j=i}^{n} \sum_{k=i}^{n} p_j p_k e^{\lambda_i (T_j - T_k)}, \quad (18)$$

from which the λ_i may be determined. For a given λ_i, we find the coefficients A_i and B_i as follows:

$$A_i = \frac{\sum_{j=1}^{n} p_j e^{\lambda_i (T_j - 2T_i)}}{p_0 - \lambda_i - \sum_{j=1}^{i} p_j e^{\lambda_i T_j} + \sum_{j=i}^{n} p_j e^{\lambda_i (T_j - 2T_i)}}, \quad (19)$$

$$B_i = \frac{p_0 - \lambda_i - \sum_{j=1}^{i} p_j e^{\lambda_i T_j}}{p_0 - \lambda_i - \sum_{j=1}^{i} p_j e^{\lambda_i T_j} + \sum_{j=i}^{n} p_j e^{\lambda_i (T_j - 2T_i)}} \quad (1 \leqslant i \leqslant n). \quad (20)$$

(iii) On the interval (T_{k-1}, T_k) we define $\Delta T = T_n/2 - T_k$. We use the same ansatz for the correlation function that is now defined only on the interval $(T_{k-1}, T_{k-1} + \Delta T)$. We repeat the step (ii) and finally obtain $C(t)$ for $0 < t < T_n/2$. Once $C(t - T_n)$ is known we can calculate $C(t)$ at all $t > T_n$,

$$C(t) = e^{-p_0 t}\left[C(T_n) + \int_{T_n}^{t} e^{p_0(t'-t)} \sum_{j=1}^{n} p_j C(t' - T_j) dt'\right]. \quad (21)$$

It is noted that when $n=1$ our results reduce to that obtained in Ref. [9]. The solution structure depends now on the time-delay distribution $\{T_i\}$ and the sign of the delay feedbacks $\{\epsilon_i\}$ as well.

The power spectrum can be determined from the correlation function by the following definition:

$$S(\omega) = 2 \ \mathrm{Re}\ L(\omega),$$

$$L(\omega) = \int_0^{\infty} C(t) e^{i\omega t} dt. \quad (22)$$

By substituting this definition into Eq. (12) we find

$$-C(0) - i\omega L(\omega) = -p_0 L(\omega) + \sum_{j=1}^{n} p_j e^{i\omega T_j}\left[L(\omega) + \sum_{k=0}^{j-1} L_k(\omega) e^{-i\omega T_k}\right], \quad (23)$$

where $L_k(\omega) = \int_0^{T_{k+1}-T_k} c(\tau)\exp(-i\omega\tau) d\tau$. Remember that $C(0)=1$ and $T_0=0$, we obtain

$$L(\omega) = \frac{1 + \sum_{j=1}^{n} p_j e^{i\omega T_j} \sum_{k=0}^{j-1} e^{-i\omega T_k} L_k(\omega)}{p_0 - i\omega - \sum_{j=1}^{n} p_j e^{i\omega T_j}}. \quad (24)$$

We now calculate the linear response of the delayed system to a weak periodic signal. Following the similar argument of Ref. [15], we assume that the transition rates (5) are modulated with a frequency Ω according to the Arrhenius rate law,

$$W'_{\pm}(t) = W_{\pm}(t) e^{\pm \gamma(t)}, \quad (25)$$

where $\gamma(t) = \mu D^{-1}\cos(\Omega t + \phi)$. Define $\sigma(t) = n_+(t) - n_-(t)$, and the time evolution of $\sigma(t)$ is determined by

$$\frac{d\sigma}{dt} = W_- e^{-\gamma(t)} - W_+ e^{\gamma(t)} - (W_+ e^{\gamma(t)} + W_- e^{-\gamma(t)})\sigma. \quad (26)$$

Taking only the linear terms in $W_{\pm}(t)$, we find

$$W_{\pm}(t) = p_0 \pm \sum_{j=1}^{n} p_j \sigma(t - T_j). \quad (27)$$

Now suppose that $\sigma = \sigma_0 + \mu D^{-1}\sigma_1$ and $\mu \ll 1$, the linearized equation for σ reads

$$\frac{d\sigma_1}{dt} = -p_0\sigma_1(t) - \sum_{i=1}^{n} p_i \sigma_1(t - T_i) - p_0\cos(\Omega t + \phi). \quad (28)$$

Using the ansatz $\sigma_1(t) = A e^{i(\Omega t + \phi)}$, we find

$$A = -\frac{p_0}{p_0 + \sum_{j=1}^{n} p_j e^{-i\Omega T_j} + i\Omega} \quad (29)$$

and the solution to Eq. (28) is given by

$$\sigma_1(t) = -\mathrm{Re}\frac{p_0 e^{i(\Omega + \phi)}}{p_0 + \sum_{j=1}^{n} p_j e^{-i\Omega T_j} + i\Omega}, \quad (30)$$

which is the periodic part at the frequency Ω in the dichotomic process under study. Thus, the linear response defined by $\eta = |A|^2/D^2$, is given by

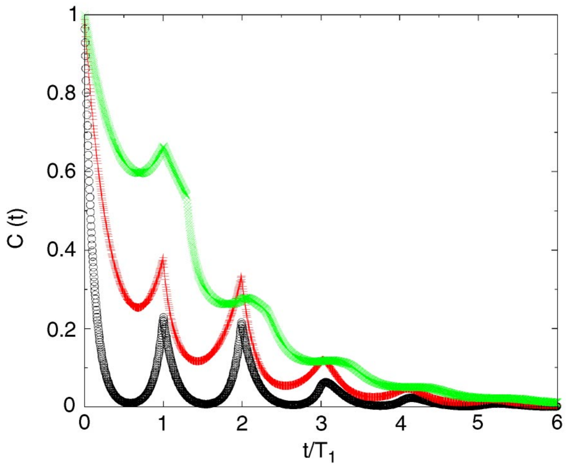

FIG. 1. Comparison of the autocorrelation function in the two-state model with the single and double time delays. The noise intensity is $D=0.1$, the feedback coupling strengths are $\epsilon=0.05$ for single time delay, and $\epsilon_1=0.05$ and $\epsilon_2=0.01$ for double time delays, respectively. The delay times are $T_1=250$ and $T_2=0$ (circles, the single time delay), $T_2=490$ (pluses), and $T_2=650$ (crosses).

$$\eta=\frac{1}{2D^2}\frac{p_0^2}{\left|2p_0+2\sum_{j=1}^{n}p_je^{-i\Omega T_j}+i\Omega\right|^2}. \qquad (31)$$

To explain the effects of multiple time delays, we compare the autocorrelation functions for the single and double time delays. From Fig. 1 we see that the oscillatory character of the correlation function becomes less pronounced as the second time delay is increased. It is expected that when the time delay is sufficiently large, the exponential decay is recovered. Another interesting observation is that within the approximation taken in the present work, the multiple time delay do not generate multiple-periodic behavior of the correlation function, though the peak-to-peak period in the correlation function seems to increase slightly as the time delay is increased. We also studied the linear response for those two cases. For the large double delay times, η decays in an oscillatory manner, with deformed wave forms, but with the same frequency as that of the single time delay. These results strongly suggest that large multiple time delay may substantially change the resonance properties of the bistable noisy system, and deserves more detailed investigation.

In summary, we have studied the resonance behavior in a noise-driven nonlinear bistable dynamical system with multiple time delays, in the regime of weak stochastic perturbation and small amplitudes of delayed feedbacks. We have derived analytical formulas of autocorrelation function, power spectrum, and linear response for the corresponding two-state system. Our results may be relevant for a class of stochastic systems with memory that is described by a distributed time delays, and may be extended to a more general coupled multistable dynamical systems where the collective property of the system depends on the past of their constituents through some memory kernels which represent the time-delayed interaction configurations. It should be noted that the analytical results presented in this work are valid only within the approximations used in the derivation. Nevertheless, our results may serve as a guide in many complex situations, at least qualitatively.

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[1] L. Gammaitoni, P. Hanggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).
[2] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **71**, 3625 (1993).
[3] L. Gammaitoni, F. Marchesoni, and S. Santucci, Phys. Rev. Lett. **74**, 1052 (1995).
[4] A. Longtin, A. R. Bulsara, and F. Moss, Phys. Rev. Lett. **67**, 656 (1991).
[5] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).
[6] S. Guillouzic, I. L'Heureux, and A. Longtin, Phys. Rev. E **59**, 3970 (1999); **61**, 4906 (2000).
[7] T. Ohira, Phys. Rev. E **55**, R1255 (1997).
[8] T. Ohira and Y. Sato, Phys. Rev. Lett. **82**, 2811 (1999).
[9] L. S. Tsimring and A. Pikovsky, Phys. Rev. Lett. **87**, 250602 (2001).
[10] C. Masoller, Phys. Rev. Lett. **90**, 020601 (2003); **88**, 034102 (2002).
[11] J. Foss, A. Longtin, B. Mansour, and J. Milton, Phys. Rev. Lett. **76**, 708 (1996).
[12] C. W. Eurich, K. Pawelzik, U. Ernst, J. D. Cowan, and J. Milton, Phys. Rev. Lett. **82**, 1594 (1999).
[13] C. Mackey and L. Glass, Science **197**, 287 (1977).
[14] C. M. Marcus and R. M. Westervelt, Phys. Rev. A **39**, 347 (1989).
[15] P. C. Bressoff and S. Coombes, Phys. Rev. Lett. **78**, 4665 (1997).
[16] S. Kim, S. H. Park, and C. S. Ryn, Phys. Rev. Lett. **82**, 1620 (1999).
[17] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).